On the recursive sequence $x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}$

C. J. Schinas, G. Papaschinopoulos, G. Stefanidou,
Democritus University of Thrace
School of Engineering
67100 Xanthi
Greece

Abstract

This paper studies the behavior of positive solutions of the difference equation

$$x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, ...,$$

where $A, p, q \in (0, \infty)$ and $x_{-1}, x_0 \in (0, \infty)$.

Keywords: Difference equation, boundedness, persistence, attractivity, asymptotic stability, periodicity.

1 Introduction

Difference equations have been applied in several mathematical models in biology, economics, genetics, population dynamics etc. For this reason, there exists an increasing interest in studying difference equations (see [1]-[28] and the references cited therein).

The investigation of positive solutions of the following equation

$$x_n = A + \frac{x_{n-k}^p}{x_{n-m}^q}, \quad n = 0, 1, ...,$$

where $A, p, q \in [0, \infty)$ and $k, m \in N$, $k \neq m$, was proposed by Stević at numerous conferences. For some results in the area see, for example, [3], [4], [5], [8], [11], [12], [19], [22], [24], [25], [28].

In [22] the author studied the boundedness, the global attractivity, the oscillatory behavior and the periodicity of the positive solutions of the equation

$$x_{n+1} = a + \frac{x_{n-1}^p}{x_n^p}, \quad n = 0, 1, \dots ,$$

where a, p are positive constants and the initial conditions x_{-1}, x_0 are positive numbers (see also [5] for more results on this equation).

In [11] the authors obtained boundedness, persistence, global attractivity and periodicity results for the positive solutions of the difference equation

$$x_{n+1} = a + \frac{x_{n-1}}{x_n^p}, \quad n = 0, 1, \dots ,$$

where a, p are positive constants and the initial conditions x_{-1}, x_0 are positive numbers.

Motivating by the above papers, we study now the boundedness, the persistence, the existence of unbounded solutions, the attractivity, the stability of the positive solutions and the period two solutions of the difference equation

$$x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \dots , \tag{1.1}$$

where A, p, q are positive constants and the initial values x_{-1}, x_0 are positive real numbers.

Finally equations, closely related to Eq. (1.1), are considered in [1]-[11], [14], [16]-[23], [26], [27], and the references cited therein.

2 Boundedness and persistence

The following result is essentially proved in [22]. Hence, we omit its proof.

Proposition 2.1 *If*

$$0 < p < 1, \tag{2.1}$$

then every positive solution of Eq.(1.1) is bounded and persists.

In the next proposition we obtain sufficient conditions for the existence of unbounded solutions of Eq.(1.1).

Proposition 2.2 *If*

$$p > 1 \tag{2.2}$$

then there exist unbounded solutions of Eq.(1.1).

Proof Let x_n be a solution of (1.1) with initial values x_{-1}, x_0 such that

$$x_{-1} > \max\left\{(A+1)^{\frac{p}{q}},\ \ (A+1)^{\frac{q}{p-1}}\right\}, \quad x_0 < A+1. \tag{2.3}$$

Then from (1.1), (2.2), (2.3) we have,

$$\begin{aligned} x_1 &= A + \frac{x_{-1}^p}{x_0^q} > A + \frac{x_{-1}^p}{(A+1)^q} - x_{-1} + x_{-1} \\ &= A + x_{-1}\Big(\frac{x_{-1}^{p-1}}{(A+1)^q} - 1\Big) + x_{-1} > A + x_{-1}. \end{aligned} \tag{2.4}$$

$$x_2 = A + \frac{x_0^p}{x_1^q} < A + \frac{(A+1)^p}{x_{-1}^q} < A+1. \tag{2.5}$$

Moreover from (1.1), (2.3) we have

$$x_1 = A + \frac{x_{-1}^p}{x_0^q} > A + \frac{(A+1)^{\frac{qp}{p-1}}}{(A+1)^q} = A + (A+1)^{\frac{q}{p-1}} > (A+1)^{\frac{q}{p-1}}. \tag{2.6}$$

Then using (1.1), (2.3)-(2.6) and arguing as above we get

$$x_3 = A + \frac{x_1^p}{x_2^q} > A + \frac{x_1^p}{(A+1)^q} - x_1 + x_1 > A + x_1.$$

$$x_4 = A + \frac{x_2^p}{x_3^q} < A + \frac{(A+1)^p}{x_{-1}^q} < A+1.$$

Therefore working inductively we can prove that for $n = 0, 1, ...$

$$x_{2n+1} > A + x_{2n-1}, \quad x_{2n} < A+1$$

which implies that

$$\lim_{n\to\infty} x_{2n+1} = \infty.$$

So x_n is unbounded. This completes the proof of the proposition.

3 Attractivity and Stability

In the following proposition we prove the existence of a positive equilibrium.

Proposition 3.1 *If either*

$$0 < q < p < 1 \tag{3.1}$$

or

$$0 < p < q \tag{3.2}$$

hold then Eq.(1.1) has a unique positive equilibrium $\bar{x}$.

Proof. A point $\bar{x} \in I\!R$ will be an equilibrium of Eq.(1.1) if and only if satisfies the following equation

$$F(x) = x^{p-q} - x + A = 0.$$

Suppose that (3.1) is satisfied. Since (3.1) holds and

$$F'(x) = (p-q)x^{p-q-1} - 1 \tag{3.3}$$

we have that F is increasing in $[0, (p-q)^{\frac{1}{-p+q+1}}]$ and F is decreasing in $[(p-q)^{\frac{1}{-p+q+1}}, \infty)$. Moreover $F(0) = A > 0$ and

$$\lim_{x \to \infty} F(x) = -\infty. \tag{3.4}$$

So if (3.1) holds we get that Eq.(1.1) has a unique equilibrium $\bar{x}$ in $(0, \infty)$.

Suppose now that (3.2) holds. We observe that $F(1) = A > 0$ and since from (3.2), (3.3) $F'(x) < 0$, we have that F is decreasing in $(0, \infty)$. Thus from (3.4) we obtain that Eq.(1.1) has a unique equilibrium $\bar{x}$ in $(0, \infty)$. The proof is complete.

In the sequel, we study the global asymptotic stability of the positive solutions of Eq.(1.1).

Proposition 3.2 *Consider Eq.(1.1). Suppose that either*

$$0 < p < 1 < q, \quad A > (p+q-1)^{\frac{1}{q-p+1}} \tag{3.5}$$

or (3.1) and

$$0 < p + q \leq 1. \tag{3.6}$$

hold. Then the unique positive equilibrium of Eq.(1.1) is globally asymptotically stable.

Proof. First we prove that every positive solution of Eq.(1.1) tends to the unique positive equilibrium $\bar{x}$ of Eq.(1.1).

Assume first that (3.5) are satisfied. Let x_n be a positive solution of Eq.(1.1). From (3.5) and Proposition 2.1 we have

$$0 < l = \liminf_{n\to\infty} x_n, \quad L = \limsup_{n\to\infty} x_n < \infty. \tag{3.7}$$

Then from (1.1) and (3.7) we get,

$$L \leq A + \frac{L^p}{l^q}, \quad l \geq A + \frac{l^p}{L^q}$$

and so

$$Ll^q \leq Al^q + L^p, \quad lL^q \geq AL^q + l^p.$$

Thus,

$$AL^q l^{q-1} + l^p l^{q-1} \leq Al^q L^{q-1} + L^p L^{q-1}.$$

This implies that

$$AL^{q-1} l^{q-1}(L - l) \leq L^{p+q-1} - l^{p+q-1}. \tag{3.8}$$

Suppose for a while that $p + q - 2 \geq 0$. We shall prove that $l = L$. Suppose on the contrary that $l < L$. If we consider the function x^{p+q-1} then there exists a $c \in (l, L)$ such that

$$\frac{L^{p+q-1} - l^{p+q-1}}{L - l} = (p + q - 1)c^{p+q-2} \leq (p + q - 1)L^{p+q-2}. \tag{3.9}$$

Then from (3.8) and (3.9) we obtain

$$AL^{q-1} l^{q-1} \leq (p + q - 1)L^{p+q-2}$$

or

$$AL^{1-p} l^{q-1} \leq p + q - 1. \tag{3.10}$$

Moreover, since from (1.1),

$$L \geq A, \quad l \geq A$$

from (3.5) and (3.10) we get

$$AA^{1-p} A^{q-1} = A^{q-p+1} \leq p + q - 1$$

which contradicts to (3.5). So $l = L$ which implies that x_n tends to the unique positive equilibrium $\bar{x}$.

Suppose that $p+q-2<0$. Then from (3.8) and arguing as above we get

$$AL^{q-1}l^{q-1} \leq (p+q-1)l^{p+q-2}.$$

Then arguing as above we can prove that x_n tends to the unique positive equilibrium $\bar{x}$.

Assume now that (3.6) holds. From (3.6) and (3.8) we obtain

$$AL^{q-1}l^{q-1}(L-l) \leq \frac{1}{L^{1-p-q}} - \frac{1}{l^{1-p-q}} = \frac{l^{1-p-q} - L^{1-p-q}}{L^{1-p-q}l^{1-p-q}} \leq 0$$

which implies that $L = l$. So every positive solution x_n of Eq.(1.1) tends to the unique positive equilibrium $\bar{x}$ of Eq.(1.1).

It remains to prove now that the unique positive equilibrium of Eq.(1.1) is locally asymptotically stable. The linearized equation about the positive equilibrium $\bar{x}$ is the following

$$y_{n+2} + q\bar{x}^{p-q-1}y_{n+1} - p\bar{x}^{p-q-1}y_n = 0. \tag{3.11}$$

Using Theorem 1.3.4. of [13] the linear equation (3.11) is asymptotically stable if and only if

$$q\bar{x}^{p-q-1} < -p\bar{x}^{p-q-1} + 1 < 2. \tag{3.12}$$

First assume that (3.5) hold. Since (3.5) hold then we obtain that

$$A > (p+q)^{\frac{p-q}{q+1-p}}(q+p-1). \tag{3.13}$$

From (3.5) and (3.13) we can easily prove that

$$F(c) > 0, \ where\ c = (p+q)^{\frac{1}{q+1-p}}. \tag{3.14}$$

Therefore

$$\bar{x} > (p+q)^{\frac{1}{q+1-p}} \tag{3.15}$$

which implies that (3.12) is true. So in this case the unique positive equilibrium $\bar{x}$ of Eq.(1.1) is locally asymptotically stable.

Finally suppose that (3.1) and (3.6) are satisfied. Then we can prove that (3.14) is satisfied and so the unique positive equilibrium $\bar{x}$ of Eq.(1.1) satisfies (3.15). Therefore (3.12) are hold. This implies that the unique positive equilibrium $\bar{x}$ of Eq.(1.1) is locally asymptotically stable. This completes the proof of the proposition.

4 Study of 2-periodic solutions

Motivated by Lemma 1 of [5], in this section we show that there is a prime two periodic solution. Moreover we find solutions of (1.1) which converge to a prime two periodic solution.

Proposition 4.1 *Consider Eq.(1.1) where*

$$0 < p < 1 < q. \tag{4.1}$$

Assume that there exists a sufficient small positive real number ϵ_1*, such that*

$$\frac{1}{(A+\epsilon_1)^{q-p}} > \epsilon_1 \tag{4.2}$$

and

$$(A+\epsilon_1)^{\frac{p}{q}}\epsilon_1^{-\frac{1}{q}} < A + \epsilon_1^{-p/q}(A+\epsilon_1)^{\frac{p^2-q^2}{q}}. \tag{4.3}$$

Then Eq.(1.1) has a periodic solution of prime period two.

Proof. Let x_n be a positive solution of Eq.(1.1). It is obvious that if

$$x_{-1} = A + \frac{x_{-1}^p}{x_0^q}, \quad x_0 = A + \frac{x_0^p}{x_{-1}^q},$$

then x_n is periodic of period two. Consider the system

$$x = A + \frac{x^p}{y^q}, \quad y = A + \frac{y^p}{x^q}, \tag{4.4}$$

Then system (4.4) is equivalent to

$$y - A - \frac{y^p}{x^q} = 0, \quad y = \frac{x^{\frac{p}{q}}}{(x-A)^{\frac{1}{q}}} \tag{4.5}$$

and so we get the equation

$$G(x) = \frac{x^{\frac{p}{q}}}{(x-A)^{\frac{1}{q}}} - A - \frac{x^{\frac{p^2-q^2}{q}}}{(x-A)^{\frac{p}{q}}} = 0. \tag{4.6}$$

We obtain

$$G(x) = \frac{1}{(x-A)^{\frac{1}{q}}}\left(x^{\frac{p}{q}} - x^{\frac{p^2-q^2}{q}}(x-A)^{\frac{1-p}{q}}\right) - A$$

and so from (4.1)

$$\lim_{x \to A+} G(x) = \infty.$$

Moreover from (4.3) we can show that

$$G(A + \epsilon_1) < 0. \tag{4.7}$$

Therefore the equation $G(x) = 0$ has a solution $\bar{x} = A + \epsilon_0$, where $0 < \epsilon_0 < \epsilon_1$, in the interval $(A, A + \epsilon_1)$. We have

$$\bar{y} = \frac{\bar{x}^{\frac{p}{q}}}{(\bar{x} - A)^{\frac{1}{q}}}.$$

We consider the function

$$H(\epsilon) = (A + \epsilon)^{p-q} - \epsilon.$$

Since form (4.1) $H'(\epsilon) = (p - q)(A + \epsilon)^{p-q-1} - 1 < 0$ we have

$$H(\epsilon_0) > H(\epsilon_1). \tag{4.8}$$

From (4.2) we have $H(\epsilon_1) > 0$, so from (4.8)

$$H(\epsilon_0) = (A + \epsilon_0)^{p-q} - \epsilon_0 > 0$$

which implies that

$$\bar{x} = A + \epsilon_0 < \frac{(A + \epsilon_0)^{\frac{p}{q}}}{\epsilon_0^{\frac{1}{q}}} = \bar{y}.$$

Hence, if $x_{-1} = \bar{x}$, $x_0 = \bar{y}$, then the solution x_n with initial values x_{-1}, x_0 is a prime 2-periodic solution.

In the sequel, we shall need the following lemmas.

Lemma 4.1 *Let $\{x_n\}$ be a solution of (1.1). Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone.*

Proof. We define the sequence $\{u_n\}$ and the function $h(x)$ as follows

$$u_n = x_n - A, \quad h(x) = x + A.$$

Then from (1.1) for $n \geq 3$ we get

$$\frac{u_n}{u_{n-2}} = \frac{(u_{n-2} + A)^p}{(u_{n-4} + A)^p} \frac{(u_{n-3} + A)^q}{(u_{n-1} + A)^q} = \frac{(h(u_{n-2}))^p}{(h(u_{n-4}))^p} \frac{(h(u_{n-3}))^q}{(h(u_{n-1}))^q}. \tag{4.9}$$

Then using (4.9) and arguing as in Lemma 2 of [5] (see also Theorem 2 in [20]) we can easily prove the lemma.

Lemma 4.2 *Consider equation (1.1) where (4.1) and (4.3) hold. Let x_n be a solution of (1.1) such that either*

$$A < x_{-1} < A + \epsilon_1, \quad x_0 > (A + \epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}} \tag{4.10}$$

or

$$A < x_0 < A + \epsilon_1, \quad x_{-1} > (A + \epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}. \tag{4.11}$$

Then if (4.10) hold we have

$$A < x_{2n-1} < A + \epsilon_1, \quad x_{2n} > (A + \epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}, \quad n = 0, 1, ... \tag{4.12}$$

and if (4.11) are satisfied we have

$$A < x_{2n} < A + \epsilon_1, \quad x_{2n-1} > (A + \epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}, \quad n = 0, 1, ... \tag{4.13}$$

Proof Suppose that (4.10) are satisfied. Then from (1.1) and (4.3) we have

$$A < x_1 = A + \frac{x_{-1}^p}{x_0^q} < A + \epsilon_1 \frac{(A + \epsilon_1)^p}{(A + \epsilon_1)^p} = A + \epsilon_1$$

and

$$x_2 = A + \frac{x_0^p}{x_1^q} > A + (A + \epsilon_1)^{\frac{p^2 - q^2}{q}} \epsilon_1^{-\frac{p}{q}} > (A + \epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}.$$

Working inductively we can easily prove relations (4.12). Similarly if (4.11) are satisfied we can prove that (4.13) hold.

Proposition 4.2 *Consider equation (1.1) where (4.1), (4.2) and (4.3) hold. Suppose also that*

$$A + \epsilon_1 < 1. \tag{4.14}$$

Then every solution x_n of (1.1) with initial values x_{-1}, x_0 which satisfy either (4.10) or (4.11), converges to a prime two periodic solution.

Proof Let x_n be a solution with initial values x_{-1}, x_0 which satisfy either (4.10) or (4.11). Using Proposition 2.1 and Lemma 4.1 we have that there exist

$$\lim_{n\to\infty} x_{2n+1} = L, \quad \lim_{n\to\infty} x_{2n} = l.$$

In addition from Lemma 4.2 we have that either L or l belongs to the interval $(A, A + \epsilon_1)$. Furthermore from Proposition 3.1 we have that equation (1.1) has a unique equilibrium $\bar{x}$ such that $1 < \bar{x} < \infty$. Therefore from

(4.14) we have that $L \neq l$. So x_n converges to a prime two period solution. This completes the proof of the proposition.

Acknowledgements The authors would like to thank the referees for their helpful suggestions.

References

[1] A.M. Amleh, D.A. Georgiou, E.A. Grove, and G. Ladas, On the recursive sequence $x_{n+1} = a + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl., 233 (1999), 790-798.

[2] K.S. Berenhaut, J.D. Foley and S. Stević, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, Proc. Amer. Math. Soc., 135 (2007), 1133-1140.

[3] K.S. Berenhaut, J.D. Foley and S. Stević, The global attractivity of the rational difference equation $y_n = A + (\frac{y_{n-k}}{y_{n-m}})^p$, Proc. Amer. Math. Soc., 136 (2008), 103-110.

[4] K.S. Berenhaut, and S. Stević, A note on positive nonoscillatory solutions of the difference equation $x_{n+1} = \alpha + \frac{x^p_{n-k}}{x^p_n}$, J. Difference Equ. Appl., 12 (5) (2006), 495-499.

[5] K.S. Berenhaut and S. Stević, The behaviour of the positive solutions of the difference equation $x_{n+1} = A + \frac{x^p_{n-2}}{x^p_{n-1}}$, J. Difference Equ. Appl., 12 (9) (2006), 909-918.

[6] L. Berg, On the asymptotics of nonlinear difference equations, Z. Anal. Anwendungen 21 (4) (2002), 1061-1074.

[7] R. DeVault, V. Kocic and D. Stutson, Global behavior of solutions of the nonlinear difference equation $x_{n+1} = p_n + \frac{x_{n-1}}{x_n}$, J. Difference Equ. Appl. Vol. 11, No. 8, (2005), 707-719.

[8] H.M. El-Owaidy, A.M. Ahmed and M.S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = a + \frac{x^p_{n-k}}{x^p_n}$, J. Appl. Math. Comput. 12(1-2), (2003), 31-37.

[9] E.A. Grove, and G. Ladas, Periodicities in Nonlinear Difference Equations Chapman & Hall/CRC, 2005.

[10] L. Gutnik and S. Stević, On the behavior of the solutions of a second order difference equation, Discrete Dyn. Nat. Soc. Vol. 2007, Article ID 27562, (2007), 14 pages.

[11] A. E. Hamza, A. Morsyb, On the recusrsive sequence $x_{n+1} = A + \frac{x_{n-1}}{x_n^k}$, Applied Mathematics Letters, 22(2009) 91-95.

[12] B. Iričanin and S. Stević, On a class of third-order nonlinear difference equations, Appl. Math. Comput., 213 (2009), 479-483.

[13] V.L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order With Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[14] M.R.S. Kulenović, G. Ladas and C.B. Overdeep, On the dynamics of $x_{n+1} = p_n + \dfrac{x_{n-1}}{x_n}$ with a period-two coefficient, J. Difference Equ. Appl., 10 (2004), 905-914.

[15] M.R.S. Kulenović and G. Ladas, Dynamics of Second Order Rational Difference Equations Chapman & Hall/CRC, 2002.

[16] G. Papaschinopoulos and C.J. Schinas, On a $(k+1)$-th order difference equation with a coeficient of period $(k+1)$, J. Difference Equ. Appl., 11 (2005), 215-225.

[17] G. Papaschinopoulos and C.J. Schinas, On a nonautonomous difference equation with bounded coefficient, J. Math. Anal., 326 (2007), 155-164.

[18] G. Papaschinopoulos, C.J. Schinas and G. Stefanidou, On a difference equation with 3-periodic coefficient, J. Difference Equ. Appl., 11 (2005), 1281-1287.

[19] G. Papaschinopoulos, C.J. Schinas and G. Stefanidou, Boundedness, periodity and stability of the difference equation $x_{n+1} = A_n + (\dfrac{x_{n-1}}{x_n})^p$, Int. J. Dyn. Syst. Differ. Equ. 1(2007), no. 2. 109-116.

[20] S. Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$ II, Dynamic. Contin. Discrete Impuls. Systems 10 a (6) (2003), 911-917.

[21] S. Stević, A note on periodic character of a difference equation, J. Difference Equ. Appl., 10(10) (2004), 929-932.

[22] S. Stević, On the recursive sequence $x_{n+1} = a + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. Comput. 18(1-2) (2005), 229-234.

[23] S. Stević, Asymptotics of some classes of higher order difference equations, Discrete Dyn. Nat. Soc. Vol. 2007, Article ID 56813, (2007) 20 pages.

[24] S. Stević, On the recursive sequence $x_{n+1} = a + \frac{x_n^p}{x_{n-1}^p}$, Discrete Dyn. Nat. Soc. Vol. 2007, Article ID 34517, (2007) 9 pages.

[25] S. Stević, On the recursive sequence $x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r}$, Discrete Dyn. Nat. Soc., Vol. 2007, Article ID 40963, (2007) 9 pages.

[26] S. Stević, On the difference equation $x_{n+1} = a + \frac{x_{n-1}}{x_n}$, Comput. Math. Appl., 56 (5) (2008), 1159-1171.

[27] S. Stević and K.S. Berenhaut, The behavior of the positive solutions of the difference equation $x_n = \frac{f(x_{n-2})}{g(x_{n-1})}$, Abstr. Appl. Anal., Vol. 2008, Article ID 53243, (2008), 9 pages.

[28] S. Stević, Boundedness character of a class of difference equations, Nonlinear Anal. TMA 70 (2009), 839-848.